

# Two-dimensional gravitational anomalies, Schwinger terms and dispersion relations\*

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## Abstract

We are dealing with two-dimensional gravitational anomalies, specifically with the Einstein anomaly and the Weyl anomaly, and we show that they are fully determined by dispersion relations independent of any renormalization procedure (or ultraviolet regularization). The origin of the anomalies is the existence of a superconvergence sum rule for the imaginary part of the relevant formfactor. In the zero mass limit the imaginary part of the formfactor approaches a  $\delta$ -function singularity at zero momentum squared, exhibiting in this way the infrared feature of the gravitational anomalies. We find an equivalence between the dispersive approach and the dimensional regularization procedure. The Schwinger terms appearing in the equal time commutators of the energy momentum tensors can be calculated by the same dispersive method. Although all computations are performed in two dimensions the method is expected to work in higher dimensions too.

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## I. INTRODUCTION

An anomaly in field theory occurs if a symmetry of the action or the corresponding conservation law, valid in the classical theory, is violated in the quantized version. This surprising feature of quantum theory discovered by Adler [1], Bell and Jackiw [2], and by Bardeen [3] in 1969 plays a fundamental role in physics (for details see Refs. [4] – [6]). Physically there is a difference between external and internal symmetries. The breakdown of an external symmetry is not dangerous for the consistency of the theory, on the contrary, it provides for instance the physical explanation for the  $\pi^0$ -decay [1], [2] or the solution to the  $U(1)$  problem in QCD [7]. On the other hand, the breakdown of an internal symmetry (i.e. gauge symmetry) leads to an inconsistency of the quantum theory, the anomalous Ward identities destroy the renormalizability of the theory [8], and also the unitarity of the  $S$ -matrix may be lost [9]. To avoid such anomalies imposes severe restrictions to the physical content of a theory. For instance, in the famous  $SU(2) \times U(1)$  standard theory for electroweak interactions one had to demand the existence of the top quark long before it was discovered.

Gravitation regarded as a gauge theory also suffers from anomalies. The gauges are the general coordinate transformations (diffeomorphisms) or the rotations in the tangent frame (Lorentz transformations) or the conformal transformations (Weyl transformations). Then in the quantum case the classical conservation law of the energy-momentum tensor can be broken – an Einstein anomaly occurs – or an antisymmetric part of the energy-momentum tensor can exist – a Lorentz anomaly occurs – or the trace of the tensor is nonvanishing – a Weyl anomaly arises (for details see e.g. Refs. [4], [10], [11]). Whereas the anomaly in the tensor trace has been found already in the seventies [12] – [18] the study of the gravitational anomalies started with the pioneering work of Alvarez-Gaumé and Witten [19] in the eighties. The anomalies have been first found within perturbation theory, they are local polynomials in the connection and curvature. The authors [19] – [21] have calculated (ultraviolet divergent) Feynman diagrams where the external gravitational field couples to a fermion loop via the energy-momentum tensor. Of course, the anomaly – reflecting the deep laws of quantum physics – must show up within other approaches too. So they have been calculated by the heat kernel method [22], [23], by Fujikawa’s path integral approach [24], [25], and by modern mathematical techniques such as differential geometry and cohomology [26] – [30] and topology (index theorems) [11], [31], [32], (for an overview see Ref. [4]).

Deeply related to anomalies are the so-called Schwinger terms [33] – [35] (for an introduction see e.g. Refs. [5], [6]). In a Yang-Mills gauge theory Schwinger terms (ST) show up as additional terms (extensions) in the canonical algebra of the equal time commutators (ETC) of the Gauss law operators (see e.g. [36] – [40]). Cohomologically they are described by the Faddeev-Mickelsson cocycle [41], [42], and geometrically they can be related to a Berry phase in the vacuum functional [43] – [46]. ST are frequently calculated within perturbation theory where the Bjorken-Johnson-Low limit [47], [48] works very well. However, the definition of a point-splitting method turns out to be more subtle and might not lead to the correct result [49], [50].

In gravitation Schwinger terms occur in the ETC of the energy-momentum tensors, c-number terms that are proportional to derivatives of the  $\delta$ -function. They can be related to the gravitational anomalies [51], and they have been calculated explicitly via the invari-

ant spectral function and via cohomological techniques [52]. Furthermore there exists an interesting relation of the ST to the curvature of the determinant line bundle [53], [54].

Our work deals with the calculation of the gravitational anomalies, specifically the Einstein anomaly and the Weyl anomaly. The Lorentz anomaly is not independent of the Einstein anomaly, both types of anomalies can be shifted into each other by a suitable counterterm [55]. For convenience we choose the case where the Lorentz anomaly is vanishing. We also calculate the gravitational Schwinger terms. The purpose of our work is to show that all these anomalous features are easily obtained by the method of dispersion relations, a less familiar but very useful approach. Some of our results we have briefly presented in Refs. [56], [57].

Already since their first introduction into quantum field theory [58] dispersion relations (DR) proved to be a very valuable tool. In connection with anomalies DR have been formulated by Dolgov and Zakharov [59] and also by Kummer [60]. In the following several authors [61] – [64] used successfully DR to determine the anomalies in the chiral current. Recently Hořejší and Schnabl [65] have applied the method to the well-known trace anomaly [66], [67] which is related to the broken dilatation (or scale) invariance. We extend in our work the method of DR to the case of pure gravitation. So we consider chiral fermion loops coupled to gravitation – for their evaluation it is enough to use gravitation as an external or nonquantized field – and we show that the gravitational anomalies and the Schwinger terms are, in fact, completely determined by dispersion relations. All calculations are performed in two dimensions.

Conceptually the DR approach is an independent and complementary view of the anomaly phenomenon as compared to the ultraviolet regularization procedures. Within DR the anomaly manifests itself as a very peculiar infrared feature of the imaginary part of the amplitude. But as we shall show there is a link between the two approaches, the DR method and the  $n$ -dimensional regularization procedure.

Our paper is organized as follows. In Section 2 we present the general structure of the considered (pseudo-) tensor amplitude and we discuss the Ward identities which we have to study. In Section 3 we introduce the dispersion relations for the relevant formfactors of the amplitude and calculate their imaginary parts via the Cutkosky rule. In order to reproduce the DR results in a definite ultraviolet regularization scheme we have worked out in detail the 't Hooft-Veltman regularization procedure in Section 4 and we have compared the several amplitudes with the results of Tomiya [52] and Alvarez-Gaumé and Witten [19]. The equivalence between the dispersive approach and the dimensional regularization procedure is given in Section 5. In Section 6 we derive the anomalous Ward identities and explain the source of the anomaly in the DR approach. From the Ward identities we deduce the linearized gravitational anomalies – the Einstein- and the Weyl anomaly – and we also determine their covariant versions, a comparison with the exact results is given. The gravitational Schwinger terms occurring in the ETC of the energy-momentum tensors we calculate in Section 7, where we adapt the dispersive approach to a method proposed by Källén [68]. Finally we summarize our main results in Section 8.

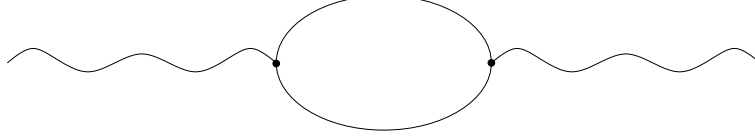


FIG. 1. Weyl fermion contribution to the graviton propagator

## II. STRUCTURE OF THE AMPLITUDE

In two dimensions the Lagrangian describing a Weyl fermion in a gravitational background field can be written as

$$\mathcal{L} = ieE^{a\mu}\bar{\psi}\gamma_a\frac{1}{2}\overleftrightarrow{D}_\mu\frac{1\pm\gamma_5}{2}\psi, \quad (2.1)$$

where  $e^a_\mu$  is the zweibein and  $E_a^\mu$  its inverse  $E_a^\mu e^a_\nu = \delta^\mu_\nu$ . The determinant of the zweibein is  $e = |\det e^a_\mu|$  and  $D_\mu = \partial_\mu + \omega_\mu$  is the covariant derivative with the spin connection  $\omega_\mu$ .

We use the following conventions in 2 dimensions:

- for the flat metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad (2.2)$$

- for the epsilon tensor

$$\varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.3)$$

- for the Dirac matrices

$$\gamma^0 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (2.4)$$

$$\gamma^1 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (2.5)$$

$$\gamma_5 = \gamma^0\gamma^1 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.6)$$

The Einstein and the Weyl anomaly are determined by the one-loop diagram in Fig.1 and it is sufficient to use the linearized gravitational field

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} + O(\kappa^2), \quad g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + O(\kappa^2), \quad (2.7)$$

$$e^a_\mu = \eta^a_\mu + \frac{1}{2}\kappa h^a_\mu + O(\kappa^2), \quad E^{a\mu} = \eta^{a\mu} - \frac{1}{2}\kappa h^{a\mu} + O(\kappa^2). \quad (2.8)$$

Since in two dimensions the spin connection  $\omega_\mu$  does not contribute (see e.g. Ref. [4]) we find the following linearized interaction Lagrangian (for convenience  $\kappa$  is absorbed into  $h^{a\mu}$ ,  $\partial_\mu^\psi$  acts only on  $\psi$ )

$$\mathcal{L}_I^{lin} = -\frac{i}{4} \left( h^{a\mu} \bar{\psi} \gamma_a \frac{1 \pm \gamma_5}{2} \overleftrightarrow{\partial}_\mu^\psi \psi + h^\mu_\mu \bar{\psi} \gamma^a \frac{1 \pm \gamma_5}{2} \overleftrightarrow{\partial}_a^\psi \psi \right) \quad (2.9)$$

and

$$\mathcal{L}_I^{lin} = -\frac{1}{2} h_{\mu\nu} T^{\mu\nu}. \quad (2.10)$$

From this expression follow the Feynman rules for the vertices in the loop diagram

$$-\frac{i}{4} (\gamma_\mu (k_1 - k_2)_\nu + \gamma_\nu (k_1 - k_2)_\mu) \frac{1 \pm \gamma_5}{2} \quad (2.11)$$

and the explicit form of the (symmetric) energy-momentum tensor

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{2} (T_a^\mu E^{a\nu} + T_a^\nu E^{a\mu}) \\ &= \frac{i}{4} \left( \bar{\psi} E^{a\nu} \gamma_a \frac{1 \pm \gamma_5}{2} \overleftrightarrow{D}^\mu \psi + \bar{\psi} E^{a\mu} \gamma_a \frac{1 \pm \gamma_5}{2} \overleftrightarrow{D}^\nu \psi \right), \end{aligned} \quad (2.12)$$

where we have dropped the terms proportional to  $g_{\mu\nu}$  as they do not contribute to the amplitude.

Then the whole amplitude is given by the two-point function

$$T_{\mu\nu\rho\sigma}(p) = i \int d^2x e^{ipx} \langle 0 | T [T_{\mu\nu}(x) T_{\rho\sigma}(0)] | 0 \rangle. \quad (2.13)$$

Due to Lorentz covariance and symmetry the general structure of the amplitude can be written in the following way

$$T_{\mu\nu\rho\sigma} = T_{\mu\nu\rho\sigma}^V + T_{\mu\nu\rho\sigma}^A \quad (2.14)$$

$$\begin{aligned} T_{\mu\nu\rho\sigma}^V(p) &= p_\mu p_\nu p_\rho p_\sigma T_1(p^2) + (p_\mu p_\nu g_{\rho\sigma} + p_\rho p_\sigma g_{\mu\nu}) T_2(p^2) \\ &\quad + (p_\mu p_\rho g_{\nu\sigma} + p_\mu p_\sigma g_{\nu\rho} + p_\nu p_\rho g_{\mu\sigma} + p_\nu p_\sigma g_{\mu\rho}) T_3(p^2) \\ &\quad + g_{\mu\nu} g_{\rho\sigma} T_4(p^2) + (g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) T_5(p^2) \end{aligned} \quad (2.15)$$

$$\begin{aligned} T_{\mu\nu\rho\sigma}^A(p) &= (\varepsilon_{\mu\tau} p^\tau p_\nu p_\rho p_\sigma + \varepsilon_{\nu\tau} p^\tau p_\mu p_\rho p_\sigma + \varepsilon_{\rho\tau} p^\tau p_\mu p_\nu p_\sigma + \varepsilon_{\sigma\tau} p^\tau p_\mu p_\nu p_\rho) T_6(p^2) \\ &\quad + (\varepsilon_{\mu\tau} p^\tau p_\nu g_{\rho\sigma} + \varepsilon_{\nu\tau} p^\tau p_\mu g_{\rho\sigma} + \varepsilon_{\rho\tau} p^\tau p_\sigma g_{\mu\nu} + \varepsilon_{\sigma\tau} p^\tau p_\rho g_{\mu\nu}) T_7(p^2) \\ &\quad + \left[ \varepsilon_{\mu\tau} p^\tau (p_\rho g_{\nu\sigma} + p_\sigma g_{\nu\rho}) + \varepsilon_{\nu\tau} p^\tau (p_\rho g_{\mu\sigma} + p_\sigma g_{\mu\rho}) \right. \\ &\quad \left. + \varepsilon_{\rho\tau} p^\tau (p_\mu g_{\nu\sigma} + p_\nu g_{\mu\sigma}) + \varepsilon_{\sigma\tau} p^\tau (p_\mu g_{\nu\rho} + p_\nu g_{\mu\rho}) \right] T_8(p^2), \end{aligned} \quad (2.16)$$

where we have separated the amplitude into its pure tensor part  $T_{\mu\nu\rho\sigma}^V$  (coming from the vector piece of the chirality projection in Eq.(2.12)) and into its pseudo-tensor part  $T_{\mu\nu\rho\sigma}^A$  (coming from the axial piece in Eq.(2.12)). The functions  $T_1(p^2), \dots, T_8(p^2)$  are the formfactors that are to be evaluated.

Classically the energy-momentum tensor has the following properties:

1.  $T_{\mu\nu} = T_{\nu\mu}$ , symmetric
2.  $\nabla^\mu T_{\mu\nu} = 0$ , conserved
3.  $T^\mu{}_\mu = 0$ , traceless

which lead to the canonical (naive) Ward identities:

1.  $T_{\mu\nu\rho\sigma}(p) = T_{\nu\mu\rho\sigma}(p)$
2.  $p^\mu T_{\mu\nu\rho\sigma}(p) = 0$
3.  $g^{\mu\nu} T_{\mu\nu\rho\sigma}(p) = 0$ .

We are interested in the pure Einstein anomaly therefore we demand the quantized energy-momentum tensor to be symmetric. This is always possible due to the Bardeen–Zumino theorem [55] which states that the gravitational anomaly can be shifted from pure Lorentz type to pure Einstein type, and vice versa. (We disregard here Leutwyler’s point of view [22,23] who emphasizes his preference for the Lorentz anomaly). Thus the symmetry property 1.) of the amplitude is fulfilled and an other symmetry  $T_{\mu\nu\rho\sigma}(p) = T_{\rho\sigma\mu\nu}(p)$  is trivially satisfied. However, the naive Ward identities 2.) and 3.) need not be satisfied, they can be broken by the Einstein and Weyl anomalies respectively.

The canonical Ward identities we re-express by the formfactors

$$p^\mu T_{\mu\nu\rho\sigma}^V(p) = p_\nu p_\rho p_\sigma (p^2 T_1 + T_2 + 2T_3) + p_\nu g_{\rho\sigma} (p^2 T_2 + T_4) + (p_\rho g_{\nu\sigma} + p_\sigma g_{\nu\rho}) (p^2 T_3 + T_5) \quad (2.17)$$

$$p^\mu T_{\mu\nu\rho\sigma}^A(p) = \varepsilon_{\nu\tau} p^\tau [p_\rho p_\sigma (p^2 T_6 + 2T_8) + g_{\rho\sigma} p^2 T_7] + \varepsilon_{\rho\tau} p^\tau [p_\nu p_\sigma (p^2 T_6 + T_7 + T_8) + g_{\nu\sigma} p^2 T_8] + \varepsilon_{\sigma\tau} p^\tau [p_\nu p_\rho (p^2 T_6 + T_7 + T_8) + g_{\nu\rho} p^2 T_8] \quad (2.18)$$

and

$$g^{\mu\nu} T_{\mu\nu\rho\sigma}^V(p) = p_\rho p_\sigma (p^2 T_1 + 2\omega T_2 + 4T_3) + g_{\rho\sigma} (p^2 T_2 + 2\omega T_4 + 2T_5) \quad (2.19)$$

$$g^{\mu\nu} T_{\mu\nu\rho\sigma}^A(p) = (\varepsilon_{\rho\tau} p^\tau p_\sigma + \varepsilon_{\sigma\tau} p^\tau p_\rho) (p^2 T_6 + 2\omega T_7 + 4T_8), \quad (2.20)$$

where  $n = 2\omega$  is the dimension. We call from now on Ward identity (WI) the property 2.) and trace identity (TI) the property 3.). For the pure tensor part of the amplitude the WI may be written as

$$p^2 T_1 + T_2 + 2T_3 = 0 \quad (2.21)$$

$$p^2 T_2 + T_4 = 0 \quad (2.22)$$

$$p^2 T_3 + T_5 = 0, \quad (2.23)$$

and the TI as

$$p^2 T_1 + 2\omega T_2 + 4T_3 = 0 \quad (2.24)$$

$$p^2 T_2 + 2\omega T_4 + 2T_5 = 0. \quad (2.25)$$

Of course, relations (2.25) and (2.24) are not independent of each other, (2.25) follows from (2.24) and (2.21) – (2.23).

In the following we shall use a renormalization procedure which keeps the WI in the pure tensor part (2.21) – (2.23) so that the anomaly occurs only in the axial pieces – representing the pseudotensor part of the amplitude.

Calculating the amplitude with massive fermions (the loop in Fig. 1) with help of the Feynman rules gives

$$T_{\mu\nu\rho\sigma}(p) = -\frac{i}{16}Tr \int \frac{d^2\omega k}{(2\pi)^{2\omega}} \left\{ [\gamma_\mu(p+2k)_\nu + \gamma_\nu(p+2k)_\mu] \frac{1 \pm \gamma_5}{2} \right. \\ \left. \times \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2 + i\varepsilon} [\gamma_\rho(p+2k)_\sigma + \gamma_\sigma(p+2k)_\rho] \frac{1 \pm \gamma_5}{2} \frac{\not{k} + m}{k^2 - m^2 + i\varepsilon} \right\}. \quad (2.26)$$

For convenience we split the pure tensor piece of the loop in the following way

$$T_{\mu\nu\rho\sigma}^V(p) = \frac{1}{2}T_{\mu\nu\rho\sigma}^{pv}(p) - T_{\mu\nu\rho\sigma}^{dv}(p), \quad (2.27)$$

where  $T_{\mu\nu\rho\sigma}^{pv}$  represents the loop with the identity instead of the chirality projectors and  $T_{\mu\nu\rho\sigma}^{dv}$  denotes the part proportional to  $m^2$ . We separate the 'no interchange' amplitudes as

$$T_{\mu\nu\rho\sigma}^V = T_{\mu\nu\rho\sigma}^{ni} + T_{\nu\mu\rho\sigma}^{ni} + T_{\mu\nu\sigma\rho}^{ni} + T_{\nu\mu\sigma\rho}^{ni}. \quad (2.28)$$

Finally, the axial part of the amplitude is connected to the vector part due to relation

$$\gamma_\mu \gamma_5 = -\varepsilon_{\mu\nu} \gamma^\nu \quad (2.29)$$

(valid only in 2 dimensions for our conventions (2.2) - (2.6)). A symmetric decomposition turns out to be useful

$$T_{\mu\nu\rho\sigma}^A(p) = \mp \left\{ \frac{1}{2}(\varepsilon_\mu^\tau T_{\tau\nu\rho\sigma}^{ni} + \varepsilon_\rho^\tau T_{\mu\nu\tau\sigma}^{ni}) + \frac{1}{2}(\varepsilon_\nu^\tau T_{\tau\mu\rho\sigma}^{ni} + \varepsilon_\rho^\tau T_{\nu\mu\tau\sigma}^{ni}) \right. \\ \left. + \frac{1}{2}(\varepsilon_\mu^\tau T_{\tau\nu\sigma\rho}^{ni} + \varepsilon_\sigma^\tau T_{\mu\nu\tau\rho}^{ni}) + \frac{1}{2}(\varepsilon_\nu^\tau T_{\tau\mu\sigma\rho}^{ni} + \varepsilon_\sigma^\tau T_{\nu\mu\tau\rho}^{ni}) \right\}. \quad (2.30)$$

### III. DISPERSION RELATIONS

The formfactors of the amplitude (2.14) – (2.16) are analytic functions in the complex  $p^2 = t$  plane except a cut on the real axis starting at  $t = 4m^2$ . Due to Cauchy's theorem they can be expressed by dispersion relations which relate the real part of the amplitude to its imaginary part.

The imaginary parts of the amplitude can be easily calculated via Cutkosky's rule [69]. It states to replace in the amplitude each propagator by its discontinuity on mass shell

$$\begin{aligned}
ImT_{\mu\nu\rho\sigma}^{ni}(p) &= \frac{1}{32} \int d^2k (p+2k)_\nu (p+2k)_\sigma \\
&\times \left[ (p+k)_\mu k_\rho + (p+k)_\rho k_\mu - g_{\mu\rho} (p+k)^\lambda k_\lambda \right] \\
&\times \delta(k^2 - m^2) \delta\left((p+k)^2 - m^2\right) \theta(-k^0) \theta(k^0 + p^0)
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
ImT_{\mu\nu\rho\sigma}^{dv,ni}(p) &= \frac{1}{32} m^2 g_{\mu\rho} \int d^2k (p+2k)_\nu (p+2k)_\sigma \\
&\times \delta(k^2 - m^2) \delta\left((p+k)^2 - m^2\right) \theta(-k^0) \theta(k^0 + p^0).
\end{aligned} \tag{3.2}$$

The explicit integration gives

$$\begin{aligned}
ImT_{\mu\nu\rho\sigma}^{ni}(p) &= \frac{1}{32} J_0 \left\{ \left[ -2 \frac{m^2}{p^2} + 8 \frac{m^4}{(p^2)^2} \right] p_\mu p_\nu p_\rho p_\sigma \right. \\
&+ \left[ -\frac{1}{6} p^2 + \frac{4}{3} m^2 - \frac{8}{3} \frac{m^4}{p^2} \right] (p_\mu p_\nu g_{\rho\sigma} + p_\mu p_\sigma g_{\nu\rho} + p_\nu p_\rho g_{\mu\sigma} + p_\rho p_\sigma g_{\mu\nu}) \\
&+ \left[ \frac{1}{3} p^2 - \frac{2}{3} m^2 - \frac{8}{3} \frac{m^4}{p^2} \right] p_\mu p_\rho g_{\nu\sigma} + \left[ \frac{1}{3} p^2 - \frac{5}{3} m^2 + \frac{4}{3} \frac{m^4}{p^2} \right] p_\nu p_\sigma g_{\mu\rho} \\
&+ \left[ \frac{1}{6} (p^2)^2 - \frac{4}{3} p^2 m^2 + \frac{8}{3} m^4 \right] (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\sigma} g_{\nu\rho}) \\
&\left. + \left[ -\frac{1}{3} (p^2)^2 + \frac{5}{3} p^2 m^2 - \frac{4}{3} m^4 \right] g_{\mu\rho} g_{\nu\sigma} \right\}
\end{aligned} \tag{3.3}$$

$$ImT_{\mu\nu\rho\sigma}^{dv,ni}(p) = \frac{1}{32} J_0 m^2 g_{\mu\rho} \left\{ (-p^2 + 4m^2) g_{\nu\sigma} + \left( 1 - 4 \frac{m^2}{p^2} \right) p_\nu p_\sigma \right\}, \tag{3.4}$$

with the threshold function

$$J_0 = \frac{1}{p^2} \left( 1 - \frac{4m^2}{p^2} \right)^{-1/2} \theta(p^2 - 4m^2), \tag{3.5}$$

from which we quickly find all imaginary parts of the formfactors in the total amplitude  $T_{\mu\nu\rho\sigma}$ :

$$ImT_1(p^2) = -\frac{1}{4} J_0 \frac{m^2}{p^2} \left( 1 - 4 \frac{m^2}{p^2} \right) \tag{3.6}$$

$$ImT_2(p^2) = -\frac{1}{48} J_0 p^2 \left( 1 - 8 \frac{m^2}{p^2} + 16 \frac{m^4}{p^4} \right) \tag{3.7}$$

$$ImT_3(p^2) = \frac{1}{96} J_0 p^2 \left( 1 + \frac{m^2}{p^2} - 20 \frac{m^4}{p^4} \right) \tag{3.8}$$

$$ImT_4(p^2) = \frac{1}{48} J_0 p^4 \left( 1 - 8 \frac{m^2}{p^2} + 16 \frac{m^4}{p^4} \right) \tag{3.9}$$

$$ImT_5(p^2) = -\frac{1}{96} J_0 p^4 \left( 1 - 2 \frac{m^2}{p^2} - 8 \frac{m^4}{p^4} \right) \tag{3.10}$$



$T_1, T_6, A_1$	unsubtracted dispersion relation
$T_2, T_3, T_7, T_8, A_2, A_3$	once subtracted dispersion relation
$T_4, T_5, A_4, A_5$	twice subtracted dispersion relation

TABLE I. The formfactors and the used type of dispersion relations

$$ImT_6(p^2) = \pm \frac{1}{16} J_0 \frac{m^2}{p^2} \left( 1 - 4 \frac{m^2}{p^2} \right) \quad (3.11)$$

$$ImT_7(p^2) = \pm \frac{1}{192} J_0 p^2 \left( 1 - 8 \frac{m^2}{p^2} + 16 \frac{m^4}{p^4} \right) \quad (3.12)$$

$$ImT_8(p^2) = \mp \frac{1}{384} J_0 p^2 \left( 1 + 4 \frac{m^2}{p^2} - 32 \frac{m^4}{p^4} \right). \quad (3.13)$$

Considering in addition the amplitude  $T_{\mu\nu\rho\sigma}^{pv}$  we have the following imaginary parts of the formfactors which we denote by:

$$ImA_1(p^2) = -\frac{1}{2} J_0 \frac{m^2}{p^2} \left( 1 - 4 \frac{m^2}{p^2} \right) \quad (3.14)$$

$$ImA_2(p^2) = -\frac{1}{24} J_0 p^2 \left( 1 - 8 \frac{m^2}{p^2} + 16 \frac{m^4}{p^4} \right) \quad (3.15)$$

$$ImA_3(p^2) = \frac{1}{48} J_0 p^2 \left( 1 + 4 \frac{m^2}{p^2} - 32 \frac{m^4}{p^4} \right) \quad (3.16)$$

$$ImA_4(p^2) = \frac{1}{24} J_0 p^4 \left( 1 - 8 \frac{m^2}{p^2} + 16 \frac{m^4}{p^4} \right) \quad (3.17)$$

$$ImA_5(p^2) = -\frac{1}{48} J_0 p^4 \left( 1 + 4 \frac{m^2}{p^2} - 32 \frac{m^4}{p^4} \right). \quad (3.18)$$

Clearly, the imaginary parts (3.14) – (3.18) of the amplitude  $T_{\mu\nu\rho\sigma}^{pv}$  satisfies the WI (2.21)–(2.23) with  $T_i \rightarrow ImA_i(p^2)$ , and the subtraction procedure we choose in the following keeps this property for the entire formfactors  $A_i(p^2)$ .

Now we start with an unsubtracted dispersion relation (DR) for the formfactors

$$T(p^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t - p^2} ImT(t) \quad (3.19)$$

and we observe that, for instance, the integral for  $T_1(p^2)$  is convergent whereas for  $T_2(p^2)$  it is logarithmically divergent and needs to be subtracted once, and for  $T_4(p^2)$  it is linearly divergent and needs to be subtracted twice. We can infer already from the  $p^2 = t$  behaviour of the imaginary parts which kind of dispersion relation we have to use, see Table I.

So for the formfactors  $T_1, T_6, A_1$  an unsubtracted DR is sufficient and we get

$$T_1(p^2) = \mp 4T_6(p^2) = \frac{1}{2} A_1(p^2)$$

$$\begin{aligned}
&= -\frac{1}{4\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{m^2}{t^2} \left(1 - \frac{4m^2}{t}\right)^{\frac{1}{2}} \\
&= \frac{1}{p^2} \left[ \frac{1}{24\pi} - \frac{1}{2\pi} \frac{m^2}{p^2} + \frac{1}{2\pi} \frac{m^2}{p^2} a(p^2) \right]
\end{aligned} \tag{3.20}$$

with

$$a(p^2) = \sqrt{\frac{4m^2 - p^2}{p^2}} \arctan \sqrt{\frac{p^2}{4m^2 - p^2}} \quad . \tag{3.21}$$

A once subtracted DR defined by

$$T^R(p^2) = T(p^2) - T(0) = \frac{p^2}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{1}{t} \text{Im}T(t) \tag{3.22}$$

we use for the following formfactors (see Table 1)

$$\begin{aligned}
T_2^R(p^2) &= \mp 4T_7^R(p^2) = \frac{1}{2} A_2^R(p^2) \\
&= \frac{p^2}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{1}{t^2} \left(1 - \frac{4m^2}{t}\right)^{-\frac{1}{2}} \left(-\frac{1}{48}t + \frac{1}{6}m^2 - \frac{1}{3} \frac{m^4}{t}\right) \\
&= -\frac{1}{18\pi} + \frac{1}{6\pi} \frac{m^2}{p^2} + \frac{1}{24\pi} \left(1 - 4 \frac{m^2}{p^2}\right) a(p^2)
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
T_3^R(p^2) &= \frac{p^2}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{1}{t^2} \left(1 - \frac{4m^2}{t}\right)^{-\frac{1}{2}} \left(\frac{1}{96}t + \frac{1}{96}m^2 - \frac{5}{24} \frac{m^4}{t}\right) \\
&= \frac{7}{576\pi} + \frac{5}{48\pi} \frac{m^2}{p^2} - \frac{1}{48\pi} \left(1 + 5 \frac{m^2}{p^2}\right) a(p^2)
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
A_3^R(p^2) &= \frac{p^2}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{1}{t^2} \left(1 - \frac{4m^2}{t}\right)^{-\frac{1}{2}} \left(\frac{1}{48}t + \frac{1}{12}m^2 - \frac{2}{3} \frac{m^4}{t}\right) \\
&= \frac{1}{72\pi} + \frac{1}{3\pi} \frac{m^2}{p^2} - \frac{1}{24\pi} \left(1 + 8 \frac{m^2}{p^2}\right) a(p^2)
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
T_8^R(p^2) &= \mp \frac{p^2}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{1}{t^2} \left(1 - \frac{4m^2}{t}\right)^{-\frac{1}{2}} \left(\frac{1}{384}t + \frac{1}{96}m^2 - \frac{1}{12} \frac{m^4}{t}\right) \\
&= \mp \frac{1}{576\pi} \mp \frac{1}{24\pi} \frac{m^2}{p^2} \pm \frac{1}{192\pi} \left(1 + 8 \frac{m^2}{p^2}\right) a(p^2) .
\end{aligned} \tag{3.26}$$

For the remaining formfactors a twice subtracted DR defined by

$$T^R(p^2) = T(p^2) - T(0) - p^2 \left. \frac{d}{dp^2} T(p^2) \right|_{p^2=0} = \frac{p^4}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{1}{t^2} \text{Im}T(t) \tag{3.27}$$

is necessary and we find

$$\begin{aligned}
T_4^R(p^2) &= \frac{1}{2}A_4^R(p^2) = \frac{p^4}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{1}{t^3} \left(1 - \frac{4m^2}{t}\right)^{-\frac{1}{2}} \left(\frac{1}{48}t^2 - \frac{1}{6}tm^2 + \frac{1}{3}m^4\right) \\
&= p^2 \left[ \frac{1}{18\pi} - \frac{1}{6\pi} \frac{m^2}{p^2} - \frac{1}{24\pi} \left(1 - 4\frac{m^2}{p^2}\right) a(p^2) \right]
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
T_5^R(p^2) &= \frac{p^4}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{1}{t^3} \left(1 - \frac{4m^2}{t}\right)^{-\frac{1}{2}} \left(-\frac{1}{96}t^2 + \frac{1}{48}tm^2 + \frac{1}{12}m^4\right) \\
&= p^2 \left[ -\frac{5}{288\pi} - \frac{1}{24\pi} \frac{m^2}{p^2} + \frac{1}{48\pi} \left(1 + 2\frac{m^2}{p^2}\right) a(p^2) \right]
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
A_5^R(p^2) &= \frac{p^4}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{1}{t^3} \left(1 - \frac{4m^2}{t}\right)^{-\frac{1}{2}} \left(-\frac{1}{48}t^2 - \frac{1}{12}tm^2 + \frac{2}{3}m^4\right) \\
&= p^2 \left[ -\frac{1}{72\pi} - \frac{1}{3\pi} \frac{m^2}{p^2} + \frac{1}{24\pi} \left(1 + 8\frac{m^2}{p^2}\right) a(p^2) \right].
\end{aligned} \tag{3.30}$$

With these explicit expressions for the formfactors we have determined the whole amplitude  $T_{\mu\nu\rho\sigma}$ , Eqs.(2.13)–(2.16), from which the correct Ward identities will follow.

#### IV. 'T HOOFT–VELTMAN REGULARIZATION

Although the method of dispersion relations appears quite different to the methods of regularizations we can reproduce its results in a definite regularization scheme, namely in the n-dimensional regularization procedure of 't Hooft–Veltman [70]. It is instructive to work it out in more detail.

We start with amplitude (2.26), calculate the  $\gamma$ -matrices and follow the standard procedure by inserting the Feynman parameter integral

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{[ax + b(1-x)]^2}. \tag{4.1}$$

Then we obtain for the ‘no interchange’ amplitudes of the pure tensor pieces

$$\begin{aligned}
T_{\mu\nu\rho\sigma}^{ni}(p) &= -i \frac{2^\omega}{32} \int_0^1 dx \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{(2l - p(1-2x))_\nu (2l - p(1-2x))_\sigma}{[l^2 - \Delta]^2} \\
&\times \left[ (l + px)_\mu (l - p(1-x))_\rho + (l + px)_\rho (l - p(1-x))_\mu - g_{\mu\rho} (l + px)^\lambda (l - p(1-x))_\lambda \right]
\end{aligned} \tag{4.2}$$

and

$$T_{\mu\nu\rho\sigma}^{dv,ni}(p) = -i \frac{2^\omega}{32} m^2 g_{\mu\rho} \int_0^1 dx \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{(2l - p(1-2x))_\nu (2l - p(1-2x))_\sigma}{[l^2 - \Delta]^2}, \tag{4.3}$$

with

$$\Delta := m^2 - p^2(1-x) + p^2(1-x)^2 = m^2 - p^2x(1-x). \quad (4.4)$$

Calculating the 't Hooft-Veltman integrals

$$P_0 = \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{1}{[l^2 - \Delta]^\alpha} = \frac{(-1)^\alpha i \Gamma(\alpha - \omega)}{(4\pi)^\omega \Gamma(\alpha)} \Delta^{\omega-\alpha} \quad (4.5)$$

$$P_1^{\mu\nu} = \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{l^\mu l^\nu}{[l^2 - \Delta]^\alpha} = \frac{\Delta}{2(\omega - \alpha + 1)} g^{\mu\nu} P_0 \quad (4.6)$$

$$\begin{aligned} P_2^{\mu\nu\rho\sigma} &= \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{l^\mu l^\nu l^\rho l^\sigma}{[l^2 - \Delta]^\alpha} \\ &= \frac{\Delta^2}{4(\omega - \alpha + 1)(\omega - \alpha + 2)} (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) P_0 \end{aligned} \quad (4.7)$$

with  $\alpha = 2$  provides the following expressions

$$\begin{aligned} T_{\mu\nu\rho\sigma}^{ni}(p) &= -i \frac{2^\omega}{32} \int_0^1 dx P_0 \left\{ -2x(1-x)(1-2x)^2 p_\mu p_\nu p_\rho p_\sigma + (1-2x)^2 \frac{\Delta}{\omega-1} \right. \\ &\quad \times (p_\mu p_\nu g_{\rho\sigma} + p_\mu p_\sigma g_{\nu\rho} + p_\nu p_\rho g_{\mu\sigma} + p_\rho p_\sigma g_{\mu\nu}) - 4x(1-x) \frac{\Delta}{\omega-1} p_\mu p_\rho g_{\nu\sigma} \\ &\quad + \left[ -(1+\omega)(1-2x)^2 \frac{\Delta}{\omega-1} + x(1-x)(1-2x)^2 p^2 \right] p_\nu p_\sigma g_{\mu\rho} \\ &\quad \left. + 2 \frac{\Delta^2}{\omega(\omega-1)} (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\sigma} g_{\nu\rho}) + \left[ -2\omega \frac{\Delta^2}{\omega(\omega-1)} + 2x(1-x)p^2 \frac{\Delta}{\omega-1} \right] g_{\mu\rho} g_{\nu\sigma} \right\} \end{aligned} \quad (4.8)$$

$$T_{\mu\nu\rho\sigma}^{dv,ni}(p) = -i \frac{2^\omega}{32} m^2 g_{\mu\rho} \int_0^1 dx P_0 \left\{ 2 \frac{\Delta}{\omega-1} g_{\nu\sigma} + (1-2x)^2 p_\nu p_\sigma \right\}. \quad (4.9)$$

These expressions we compare now with the formfactor decompositions (2.15) and (2.16) (recall Eqs.(2.28) and (2.30)) then we obtain all formfactors of the amplitude  $T_{\mu\nu\rho\sigma}$  explicitly

$$T_1(p^2) = \mp 4 \quad T_6(p^2) = i \frac{2^\omega}{4} \int_0^1 dx \, x(1-x)(1-2x)^2 P_0 \quad (4.10)$$

$$T_2(p^2) = \mp 4 \quad T_7(p^2) = -i \frac{2^\omega}{8} \int_0^1 dx (1-2x)^2 \frac{\Delta}{\omega-1} P_0 \quad (4.11)$$

$$T_3(p^2) = -i \frac{2^\omega}{32} \int_0^1 dx P_0 \left\{ -4x(1-x) \frac{\Delta}{\omega-1} + (2p^2x(1-x) - m^2)(1-2x)^2 \right\} \quad (4.12)$$

$$T_4(p^2) = -i \frac{2^\omega}{4} \int_0^1 dx \frac{\Delta^2}{\omega(\omega-1)} P_0 \quad (4.13)$$

$$T_5(p^2) = -i \frac{2^\omega}{8} \int_0^1 dx P_0 \left\{ p^2 x(1-x) \frac{\Delta}{\omega-1} - \frac{\Delta^2}{\omega} \right\} \quad (4.14)$$

$$T_8(p^2) = \pm i \frac{2^\omega}{64} \int_0^1 dx (8x^2 - 8x + 1) \frac{\Delta}{\omega-1} P_0. \quad (4.15)$$

The formfactors of the amplitude  $T_{\mu\nu\rho\sigma}^{pv}$  follow via Eq.(2.27)

$$A_1(p^2) = 2 T_1(p^2) = i \frac{2^\omega}{2} \int_0^1 dx x(1-x)(1-2x)^2 P_0 \quad (4.16)$$

$$A_2(p^2) = 2 T_2(p^2) = -i \frac{2^\omega}{4} \int_0^1 dx (1-2x)^2 \frac{\Delta}{\omega-1} P_0 \quad (4.17)$$

$$A_3(p^2) = -i \frac{2^\omega}{16} \int_0^1 dx P_0 \left\{ -4x(1-x) \frac{\Delta}{\omega-1} + 2p^2 x(1-x)(1-2x)^2 \right\} \quad (4.18)$$

$$A_4(p^2) = 2 T_4(p^2) = -i \frac{2^\omega}{2} \int_0^1 dx \frac{\Delta^2}{\omega(\omega-1)} P_0 \quad (4.19)$$

$$A_5(p^2) = -i \frac{2^\omega}{4} \int_0^1 dx P_0 \left\{ (m^2 + p^2 x(1-x)) \frac{\Delta}{\omega-1} - \frac{\Delta^2}{\omega} \right\}. \quad (4.20)$$

As we can see the formfactors  $T_1, T_6$  and  $A_1$  (Eqs.(4.10) and (4.16)) are finite in the limit  $\omega \rightarrow 1$  and explicitly we find

$$\begin{aligned} T_1(p^2) = \mp 4T_6(p^2) &= \frac{1}{2} A_1(p^2) = -\frac{1}{8\pi} \int_0^1 dx \frac{x(1-x)(1-2x)^2}{m^2 - p^2 x(1-x)} \\ &= \frac{1}{p^2} \left[ \frac{1}{24\pi} - \frac{1}{2\pi} \frac{m^2}{p^2} + \frac{1}{2\pi} \frac{m^2}{p^2} a(p^2) \right] \end{aligned} \quad (4.21)$$

with  $a(p^2)$  given by Eq.(3.21).

However, the remaining formfactors are divergent in the limit  $\omega \rightarrow 1$  and we have to renormalize them in an appropriate way. We separate the formfactors into a divergent part and a finite part

$$T_i = \frac{1}{1-\omega} T_i^{pol} + T_i^{fin}, \quad (4.22)$$

so that we can extract the finite result by a suitable prescription. The formula we need for this procedure is

$$\frac{\Gamma(1-\omega)}{(2\pi)^\omega} \Delta^{\omega-1} f(\omega) = \frac{f(1)}{2\pi} \left[ \frac{1}{1-\omega} - \ln \frac{\Delta}{2\pi} - \gamma \right] - \frac{1}{2\pi} \frac{df}{d\omega} \Big|_{\omega=1} + O(1-\omega). \quad (4.23)$$

It is the pole part  $T_i^{pol}$  of the formfactor which tells us which kind of renormalization we have to choose, for instance, for a constant a simple subtraction is sufficient, for a term proportional to  $p^2$  a double subtraction.

In this way the following formfactors are determined by a simple subtraction

$$\begin{aligned} T_2^R(p^2) &= \mp 4T_7^R(p^2) = \frac{1}{2}A_2^R(p^2) = T_2(p^2) - T_2(0) \\ &= \frac{1}{16\pi} \int_0^1 dx (1-2x)^2 \ln \frac{m^2 - p^2 x(1-x)}{m^2} \\ &= -\frac{1}{18\pi} + \frac{1}{6\pi} \frac{m^2}{p^2} + \frac{1}{24\pi} \left( 1 - 4 \frac{m^2}{p^2} \right) a(p^2) \end{aligned} \quad (4.24)$$

$$\begin{aligned} T_3^R(p^2) &= T_3(p^2) - T_3(0) = -\frac{1}{16\pi} \int_0^1 dx x(1-x) \ln \frac{m^2 - p^2 x(1-x)}{m^2} \\ &\quad + \frac{1}{64\pi} \int_0^1 dx (1-2x)^2 \frac{p^2 x(1-x)}{m^2 - p^2 x(1-x)} \\ &= \frac{7}{576\pi} + \frac{5}{48\pi} \frac{m^2}{p^2} - \frac{1}{48\pi} \left( 1 + 5 \frac{m^2}{p^2} \right) a(p^2) \end{aligned} \quad (4.25)$$

$$\begin{aligned} A_3^R(p^2) &= A_3(p^2) - A_3(0) = -\frac{1}{8\pi} \int_0^1 dx x(1-x) \ln \frac{m^2 - p^2 x(1-x)}{m^2} \\ &\quad + \frac{1}{16\pi} \int_0^1 dx (1-2x)^2 \frac{p^2 x(1-x)}{m^2 - p^2 x(1-x)} \\ &= \frac{1}{72\pi} + \frac{1}{3\pi} \frac{m^2}{p^2} - \frac{1}{24\pi} \left( 1 + 8 \frac{m^2}{p^2} \right) a(p^2) \end{aligned} \quad (4.26)$$

$$\begin{aligned} T_8^R(p^2) &= T_8(p^2) - T_8(0) = \mp \frac{1}{128\pi} \int_0^1 dx (8x^2 - 8x + 1) \ln \frac{m^2 - p^2 x(1-x)}{m^2} \\ &= \mp \frac{1}{576\pi} \mp \frac{1}{24\pi} \frac{m^2}{p^2} \pm \frac{1}{192\pi} \left( 1 + 8 \frac{m^2}{p^2} \right) a(p^2), \end{aligned} \quad (4.27)$$

whereas for the remaining formfactors a double subtraction is needed

$$\begin{aligned} T_4^R(p^2) &= \frac{1}{2}A_4^R(p^2) = T_4(p^2) - T_4(0) - p^2 \frac{d}{dp^2} T_4(p^2) \Big|_{p^2=0} \\ &= \frac{1}{8\pi} \int_0^1 dx \left[ \left( m^2 - p^2 x(1-x) \right) \ln \frac{m^2 - p^2 x(1-x)}{m^2} + p^2 x(1-x) \right] \end{aligned}$$

$$= p^2 \left[ \frac{1}{18\pi} - \frac{1}{6\pi} \frac{m^2}{p^2} - \frac{1}{24\pi} \left( 1 - 4 \frac{m^2}{p^2} \right) a(p^2) \right] \quad (4.28)$$

$$\begin{aligned} T_5^R(p^2) &= T_5(p^2) - T_5(0) - p^2 \left. \frac{d}{dp^2} T_5(p^2) \right|_{p^2=0} \\ &= \frac{1}{16\pi} \int_0^1 dx \, p^2 x(1-x) \ln \frac{m^2 - p^2 x(1-x)}{m^2} \\ &= p^2 \left[ -\frac{5}{288\pi} - \frac{1}{24\pi} \frac{m^2}{p^2} + \frac{1}{48\pi} \left( 1 + 2 \frac{m^2}{p^2} \right) a(p^2) \right] \end{aligned} \quad (4.29)$$

$$\begin{aligned} A_5^R(p^2) &= A_5(p^2) - A_5(0) - p^2 \left. \frac{d}{dp^2} A_5(p^2) \right|_{p^2=0} \\ &= \frac{1}{8\pi} \int_0^1 dx \left[ \left( m^2 + p^2 x(1-x) \right) \ln \frac{m^2 - p^2 x(1-x)}{m^2} + p^2 x(1-x) \right] \\ &= p^2 \left[ -\frac{1}{72\pi} - \frac{1}{3\pi} \frac{m^2}{p^2} + \frac{1}{24\pi} \left( 1 + 8 \frac{m^2}{p^2} \right) a(p^2) \right]. \end{aligned} \quad (4.30)$$

Of course, the formfactors  $A_i^R$  of the tensor amplitude  $T_{\mu\nu\rho\sigma}^{pv}$  satisfy the WI (2.21) – (2.23) and as the difference  $T_{\mu\nu\rho\sigma}^{dv}$  to the pure tensor amplitude is proportional to  $m^2$  (recall Eqs.(2.27),(4.9)) the amplitude  $T_{\mu\nu\rho\sigma}^V$  will also satisfy the WI (2.21) – (2.23) in the limit  $m \rightarrow 0$ .

This procedure works in complete analogy to the dispersion relation approach (see Table 1) and we clearly obtain the same results for the formfactors as one can see by comparing Eqs.(4.21)–(4.30) with (3.20)–(3.30). Even more, as we shall show in the next chapter, the two approaches are equivalent.

But before we want to consider the case of massless fermions, the limit  $m \rightarrow 0$ , in order to compare our results with the ones of other authors. Taking the limit  $m \rightarrow 0$  in the  $1/(1-\omega)$  expansion (Eq.(4.22)) of the formfactors we find the following results

$$T_1(p^2) = \mp 4 \, T_6(p^2) = \frac{1}{24\pi p^2} \quad (4.31)$$

$$\begin{aligned} T_2(p^2) &= -\frac{1}{p^2} T_4(p^2) = \mp 4 \, T_7(p^2) \\ &= \frac{1}{48\pi} \left[ \frac{1}{\omega - 1} + \gamma - \frac{8}{3} + \ln \left| \frac{p^2}{2\pi\mu^2} \right| - i\pi \right] \end{aligned} \quad (4.32)$$

$$\begin{aligned} T_3(p^2) &= -\frac{1}{p^2} T_5(p^2) = \mp 4 \, T_8(p^2) \\ &= -\frac{1}{96\pi} \left[ \frac{1}{\omega - 1} + \gamma - \frac{2}{3} + \ln \left| \frac{p^2}{2\pi\mu^2} \right| - i\pi \right]. \end{aligned} \quad (4.33)$$

which agree with the expressions we obtain when starting with  $m = 0$  from the very beginning (we also introduced here the mass  $\mu$  to keep the correct dimensionality, see Ref. [10]).

Now, for comparison Tomiya [52] in his work on the gravitational anomaly defines the amplitude with the covariant  $T^*$ -product and calculates the following formfactors

$$T_1^T(p^2) = \mp 4 T_6^T(p^2) = \frac{1}{24\pi p^2} \quad (4.34)$$

$$T_2^T(p^2) = -\frac{1}{p^2} T_4^T(p^2) = \mp 4 T_7^T(p^2) = 0 \quad (4.35)$$

$$T_3^T(p^2) = -\frac{1}{p^2} T_5^T(p^2) = \mp 4 T_8^T(p^2) = -\frac{1}{48\pi} . \quad (4.36)$$

They also satisfy the WI (2.21) – (2.23), however, they (some of them) differ from ours (4.31) – (4.33), which is not surprising as the covariant  $T^*$ -product differs from the ordinary  $T$ -product by suitable seagull terms. But as we shall show below his and our results lead to the same amplitudes and consequently to the same anomaly expression.

Using on one hand the WI (2.21) – (2.23) for the formfactors  $T_i(p^2)$  in the massless limit,  $m \rightarrow 0$ , and on the other the fact that

$$T_6 = \mp \frac{1}{4} T_1, \quad T_7 = \mp \frac{1}{4} T_2, \quad T_8 = \mp \frac{1}{4} T_3, \quad (4.37)$$

we find for the several amplitudes (2.14) – (2.16) the following expressions

$$T_{0000} = (\mp p_0 p_1^3 + p_1^4) T_1(p^2) \quad (4.38)$$

$$T_{0001} = \frac{1}{4} (4p_0 p_1^3 \mp 3p_0^2 p_1^2 \mp p_1^4) T_1(p^2) \quad (4.39)$$

$$T_{0011} = T_{0101} = \frac{1}{2} (\mp p_0^3 p_1 + 2p_0^2 p_1^2 \mp p_0 p_1^3) T_1(p^2) \quad (4.40)$$

$$T_{0111} = \frac{1}{4} (\mp p_0^4 + 4p_0^3 p_1 \mp 3p_0^2 p_1^2) T_1(p^2) \quad (4.41)$$

$$T_{1111} = (p_0^4 \mp p_0^3 p_1) T_1(p^2) . \quad (4.42)$$

As we can see all amplitudes depend only on the (convergent) formfactor  $T_1(p^2)$ , explicitly given by Eq.(4.31), and are independent of a second (divergent) formfactor, say  $T_2(p^2)$ , which one might have expected to contribute too (since we have six restrictive relations between the eight formfactors  $T_i(p^2)$ ).

So the amplitudes do not determine the formfactors uniquely! Therefore  $T_2(p^2)$  can be chosen at will and Tomiya [52] in his work renormalized the formfactors conveniently such that  $T_2(p^2)$  came out to be zero. This is in accordance with the amplitude result of Deser and Schwimmer [15], [17] who consider fields without chirality.

On the other hand, Alvarez-Gaumé and Witten [19] consider the gravitational anomaly within light-cone coordinates and use a specific regularization prescription to calculate the following amplitude

$$U(p) = i \int d^2 x e^{ipx} \langle 0 | T[T_{++}(x) T_{++}(0)] | 0 \rangle = \frac{1}{24\pi} \frac{p_+^3}{p_-} = \frac{1}{12\pi p^2} p_+^4 . \quad (4.43)$$

Their result is consistent with our calculations as we have



$$\begin{aligned}
T_{++++} &= \frac{1}{4} (T_{0000} + 4T_{0001} + 2T_{0011} + 4T_{0101} + 4T_{0111} + T_{1111}) \\
&= \frac{1}{4} \frac{1}{24\pi p^2} \left[ (1 \mp 1)p_0^4 + (4 \mp 4)p_0^3 p_1 + (6 \mp 6)p_0^2 p_1^2 + (4 \mp 4)p_0 p_1^3 + (1 \mp 1)p_1^4 \right] \\
&= \frac{1 \mp 1}{24\pi p^2} p_+^4 .
\end{aligned} \tag{4.44}$$

## V. EQUIVALENCE OF DISPERSION RELATIONS AND DIMENSIONAL REGULARIZATION

The dimensional regularization procedure is a method which satisfies the WI (2.21) – (2.23). On the other hand, within the dispersion relation approach we also have effectively renormalized in a way which keeps this WI property (2.21) – (2.23). If we had subtracted more often or at some other point than  $p^2 = 0$  we would have lost this property.

The dispersive approach and the dimensional regularization procedure are equivalent in the following sense. For the formfactors that were convergent or logarithmically divergent the corresponding integrals in both approaches do not have just identical values but we even can transform them into each other by a suitable substitution. More precise, we transform the Feynman parameter integrals over  $x$  in the dimensional regularization procedure into the corresponding dispersion integrals over  $t$ . For the formfactors that were linearly divergent the situation is as follows. Accidentally for  $T_5$  there exists such a substitution, however, not for  $T_4 = \frac{1}{2}A_4$  and  $A_5$ . But anyway, for these formfactors the integrals are identical, what is actually sufficient.

The substitutions which link the two approaches are

$$y = 1 - 2x , \quad t = \frac{4m^2}{1 - y^2} . \tag{5.1}$$

To show the equivalence we need the following relations which are valid for any function  $f$  with suitable differentiation properties

$$\frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t(t - p^2)} \left(1 - \frac{4m^2}{t}\right)^{-1/2} f(t) = \frac{1}{2\pi} \int_0^1 dx \frac{f\left(\frac{m^2}{x(1-x)}\right)}{m^2 - p^2 x(1-x)} \tag{5.2}$$

$$\int_0^1 dx f'(x) \ln \frac{m^2 - p^2 x(1-x)}{m^2} = - \int_0^1 dx f(x) \frac{p^2(-1 + 2x)}{m^2 - p^2 x(1-x)} . \tag{5.3}$$

## VI. ANOMALOUS WARD IDENTITIES AND GRAVITATIONAL ANOMALIES

Now we turn to the calculation of the Ward identities and gravitational anomalies. We consider the massless limit,  $m \rightarrow 0$ , where the formfactors  $2T_i \rightarrow A_i$  ( $i = 1, \dots, 5$ ) fulfill the WI (2.21) – (2.23). This means that the WI for the pure tensor part (2.17) is satisfied

$$p^\mu T_{\mu\nu\rho\sigma}^V(p) = 0. \quad (6.1)$$

Calculating next the WI for the pseudo tensor part (2.18) we use the formfactor identities (4.37) and we find

$$\begin{aligned} p^\mu T_{\mu\nu\rho\sigma}^A(p) &= \pm \frac{1}{4} T_2 \varepsilon_{\nu\tau} p^\tau (p_\rho p_\sigma - g_{\rho\sigma} p^2) \\ &\quad \pm \frac{1}{4} T_3 (\varepsilon_{\rho\tau} p^\tau (p_\nu p_\sigma - g_{\nu\sigma} p^2) + \varepsilon_{\sigma\tau} p^\tau (p_\nu p_\rho - g_{\nu\rho} p^2)). \end{aligned} \quad (6.2)$$

In the flat space-time limit  $g_{\mu\nu}(x) = \eta_{\mu\nu}$  we have the relation

$$\varepsilon_{\nu\tau} p^\tau (p_\rho p_\sigma - g_{\rho\sigma} p^2) = \varepsilon_{\rho\tau} p^\tau (p_\nu p_\sigma - g_{\nu\sigma} p^2) = \varepsilon_{\sigma\tau} p^\tau (p_\nu p_\rho - g_{\nu\rho} p^2) \quad (6.3)$$

and we finally obtain the anomalous result

$$p^\mu T_{\mu\nu\rho\sigma}^A(p) = \mp \frac{1}{4} p^2 T_1 \varepsilon_{\nu\tau} p^\tau (p_\rho p_\sigma - g_{\rho\sigma} p^2). \quad (6.4)$$

As we discovered already before when calculating the amplitudes, the anomalous WI depends only on the finite formfactor  $T_1 = \mp 4T_6$  with its explicit result (4.31). So the anomaly is independent of a specific renormalization procedure (as long as it preserves the WI (2.21) – (2.23)) and we clearly agree with the anomaly results of Tomiya [52] and Alvarez-Gaumé and Witten [19].

We want to emphasize that our subtraction procedure is *the* ‘natural’ choice dictated by the  $t$ –behaviour of the imaginary parts  $ImT_i(t)$  of the formfactors. Since on general grounds the imaginary parts of the amplitudes fulfill the WI (2.21) – (2.23) (in the limit  $m \rightarrow 0$ ) the ‘naturally’ chosen dispersion relations

$$\frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{p^2}{t} [t Im T_1(t) + Im T_2(t) + 2 Im T_3(t)] = 0 \quad (6.5)$$

$$\frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{p^4}{t^2} [t Im T_2(t) + Im T_4(t)] = 0 \quad (6.6)$$

$$\frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-p^2} \frac{p^4}{t^2} [t Im T_3(t) + Im T_5(t)] = 0 \quad (6.7)$$

imply the pure tensor WI (2.21) – (2.23) for the renormalized formfactors (in the limit  $m \rightarrow 0$ )

$$p^2 T_1(p^2) + T_2^R(p^2) + 2 T_3^R(p^2) = 0 \quad (6.8)$$

$$p^2 T_2^R(p^2) + T_4^R(p^2) = 0 \quad (6.9)$$

$$p^2 T_3^R(p^2) + T_5^R(p^2) = 0. \quad (6.10)$$

Therefore our subtraction procedure automatically shifts the total anomaly into the pseudotensor part of the WI (2.18).

What is the origin of the anomaly in this dispersive approach? The source of the anomaly is the existence of a superconvergence sum rule for the imaginary part of the formfactor  $T_1(p^2)$

$$\int_0^\infty dt \operatorname{Im} T_1(t) = -\frac{m^2}{4} \int_{4m^2}^\infty \frac{dt}{t^2} \left(1 - \frac{4m^2}{t}\right)^{\frac{1}{2}} = -\frac{1}{24} . \quad (6.11)$$

The anomaly originates from a  $\delta$ -function singularity of  $\operatorname{Im} T_1(t)$  when the threshold  $t = 4m^2 \rightarrow 0$  approaches zero (the infrared region)

$$\lim_{m \rightarrow 0} \operatorname{Im} T_1(t) = -\lim_{m \rightarrow 0} \frac{m^2}{4t^2} \left(1 - \frac{4m^2}{t}\right)^{\frac{1}{2}} \theta(t - 4m^2) = -\frac{1}{24} \delta(t) . \quad (6.12)$$

The limit must be performed in a distributional sense.

Then the unsubtracted dispersion relation for  $T_1(p^2)$ , Eq.(3.19), provides the result (4.31). This threshold singularity of the imaginary part of the relevant formfactor is a typical feature of the dispersion relation approach for calculating the anomaly (see e.g. Refs. [4], [62] – [65], and the Appendix B in Ref. [17]).

Next we turn to the energy-momentum tensor. From the anomalous WI

$$p^\mu T_{\mu\nu\rho\sigma}(p) = \mp \frac{1}{96\pi} \varepsilon_{\nu\tau} p^\tau (p_\rho p_\sigma - g_{\rho\sigma} p^2) \quad (6.13)$$

we can deduce the linearized consistent Einstein (or diffeomorphism) anomaly

$$\partial^\mu \langle T_{\mu\nu} \rangle = \mp \frac{1}{192\pi} \varepsilon_{\mu\nu} \partial^\mu \left( \partial_\alpha \partial_\beta h^{\alpha\beta} - \partial_\alpha \partial^\alpha h^\beta_\beta \right) . \quad (6.14)$$

For comparison we demonstrate that result (6.14) is indeed the linearization of the exact result that follows from differential geometry and topology (see for instance Ref. [4]). The exact Einstein anomaly in two dimensions is given by

$$G^E(v_\xi, \Gamma) = - \int d^2x \, e \xi_\nu \nabla_\mu \langle T^{\mu\nu} \rangle = \mp \frac{1}{96\pi} \int_{M_2} \operatorname{tr} v_\xi d\Gamma = \pm \frac{1}{96\pi} \int d^2x \, \varepsilon^{\gamma\delta} \xi^\beta \partial_\alpha \partial_\gamma \Gamma^\alpha_{\delta\beta}, \quad (6.15)$$

where

$$(v_\xi)^\beta_\alpha = \partial_\alpha \xi^\beta . \quad (6.16)$$

Therefore we get

$$\nabla^\mu \langle T_{\mu\nu} \rangle = \mp \frac{1}{96\pi} \frac{1}{e} \varepsilon^{\gamma\delta} \partial_\alpha \partial_\gamma \Gamma^\alpha_{\delta\nu} . \quad (6.17)$$

Considering the linearized gravitational field, Eq.(2.8), the Christoffel symbols become (with explicit  $\kappa$ -dependence)

$$\Gamma_{\nu\beta}^{\alpha} = \frac{1}{2}g^{\alpha\lambda}(\partial_{\beta}g_{\lambda\nu} - \partial_{\lambda}g_{\nu\beta} + \partial_{\nu}g_{\beta\lambda}) = \frac{\kappa}{2}(\partial_{\beta}h_{\nu}^{\alpha} - \partial^{\alpha}h_{\nu\beta} + \partial_{\nu}h_{\beta}^{\alpha}) + O(\kappa^2), \quad (6.18)$$

so that we find as linearization of the exact result (6.17)

$$\partial^{\mu}\langle T_{\mu\nu} \rangle = \mp \frac{1}{192\pi}\varepsilon^{\gamma\delta}\partial_{\alpha}\partial_{\gamma}(\partial_{\nu}h_{\delta}^{\alpha} - \partial^{\alpha}h_{\delta\nu}). \quad (6.19)$$

This agrees with our result (6.14) since in two dimensions we have the identity

$$\varepsilon_{\mu\beta}\partial^{\mu}(\partial_{\alpha}\partial_{\nu}h^{\alpha\beta} - \partial_{\alpha}\partial^{\alpha}h_{\nu}^{\beta}) = \varepsilon_{\mu\nu}\partial^{\mu}(\partial_{\alpha}\partial_{\beta}h^{\alpha\beta} - \partial_{\alpha}\partial^{\alpha}h_{\beta}^{\beta}). \quad (6.20)$$

Now what about the covariant Einstein anomaly? It arises when considering the covariantly transforming energy-momentum tensor  $\tilde{T}_{\mu\nu}$  which is related to our tensor definition (2.12) by the Bardeen-Zumino polynomial  $\mathcal{P}_{\mu\nu}$  [55]

$$\langle \tilde{T}_{\mu\nu} \rangle = \langle T_{\mu\nu} \rangle + \mathcal{P}_{\mu\nu}. \quad (6.21)$$

This polynomial is calculable and explicitly we find (in two dimensions) [4]

$$\nabla^{\mu}\mathcal{P}_{\mu\nu} = \mp \frac{1}{96\pi}\frac{1}{e}(\varepsilon_{\mu\nu}\nabla^{\mu}\mathcal{R} - \varepsilon^{\gamma\delta}\partial_{\alpha}\partial_{\gamma}\Gamma_{\delta\nu}^{\alpha}) \quad (6.22)$$

( $\mathcal{R}$  denotes the Ricci scalar) which leads to the covariant Einstein anomaly

$$\nabla^{\mu}\langle \tilde{T}_{\mu\nu} \rangle = \mp \frac{1}{96\pi}\frac{1}{e}\varepsilon_{\mu\nu}\nabla^{\mu}\mathcal{R}. \quad (6.23)$$

Linearizing the Ricci scalar (with explicit  $\kappa$ -dependence)

$$\mathcal{R} = \kappa(\partial_{\mu}\partial_{\nu}h^{\mu\nu} - \partial_{\mu}\partial^{\mu}h_{\nu}^{\nu}) + O(\kappa^2) \quad (6.24)$$

and using (6.18), (6.20) we get

$$\partial^{\mu}\mathcal{P}_{\mu\nu} = \mp \frac{1}{192\pi}\varepsilon_{\mu\nu}\partial^{\mu}(\partial_{\alpha}\partial_{\beta}h^{\alpha\beta} - \partial_{\alpha}\partial^{\alpha}h_{\beta}^{\beta}) \quad (6.25)$$

so that we find for the linearized covariant Einstein anomaly

$$\partial^{\mu}\langle \tilde{T}_{\mu\nu} \rangle = \mp \frac{1}{96\pi}\varepsilon_{\mu\nu}\partial^{\mu}(\partial_{\alpha}\partial_{\beta}h^{\alpha\beta} - \partial_{\alpha}\partial^{\alpha}h_{\beta}^{\beta}). \quad (6.26)$$

It is twice the linearized consistent result (6.14) as it should be.

Finally we also calculate the trace identity, Eqs.(2.19) and (2.20). Using again relations (4.37) we find

$$T_{\mu\rho\sigma}^{\mu} = (p^2T_1 + 2\omega T_2 + 4T_3) \left[ (p_{\rho}p_{\sigma} - p^2g_{\rho\sigma}) \mp \frac{1}{4}(\varepsilon_{\rho\lambda}p^{\lambda}p_{\sigma} + \varepsilon_{\sigma\lambda}p^{\lambda}p_{\rho}) \right], \quad (6.27)$$

and taking into account the WI (2.21) provides us the anomalous result

$$T_{\mu\rho\sigma}^\mu = -p^2 T_1 \left[ (p_\rho p_\sigma - p^2 g_{\rho\sigma}) \mp \frac{1}{4} (\varepsilon_{\rho\lambda} p^\lambda p_\sigma + \varepsilon_{\sigma\lambda} p^\lambda p_\rho) \right]. \quad (6.28)$$

The anomalous TI depends only on the finite formfactor  $T_1 = \mp 4T_6$  – what we know already from the previous discussion – so that it is independent of a specific renormalization procedure (as long as it preserves the WI (2.21) – (2.23)).

Inserting the formfactor, Eq.(4.31), gives

$$T_{\mu\rho\sigma}^\mu = -\frac{1}{24\pi} \left[ (p_\rho p_\sigma - p^2 g_{\rho\sigma}) \mp \frac{1}{4} (\varepsilon_{\rho\lambda} p^\lambda p_\sigma + \varepsilon_{\sigma\lambda} p^\lambda p_\rho) \right], \quad (6.29)$$

which implies the following linearization of the Weyl (or trace) anomaly

$$\langle T_\mu^\mu \rangle = \frac{1}{48\pi} \left[ (\partial_\mu \partial_\nu h^{\mu\nu} - \partial_\mu \partial^\mu h^\nu_\nu) \mp \frac{1}{2} \varepsilon_{\mu\lambda} \partial^\lambda \partial_\nu h^{\mu\nu} \right]. \quad (6.30)$$

Again, we compare our calculation with the exact result for the Weyl anomaly which is given by (see for instance Ref. [4])

$$G^W(\sigma) = \int d^2x \, e \, \sigma \langle T_\mu^\mu \rangle = \frac{1}{48\pi} \int d^2x \, e \, \sigma \left( \mathcal{R} \mp \frac{1}{2} \varepsilon^{ab} \nabla_\mu \omega_{ab}^\mu \right) \quad (6.31)$$

and consequently we have

$$\langle T_\mu^\mu \rangle = \frac{1}{48\pi} \left( \mathcal{R} \mp \frac{1}{2} \varepsilon^{ab} \nabla_\mu \omega_{ab}^\mu \right). \quad (6.32)$$

Considering the linearizations of the spin connection (with explicit  $\kappa$ -dependence)

$$\omega_{b\mu}^a = e^a_\nu \nabla_\mu E_b^\nu = e^a_\nu \partial_\mu E_b^\nu + e^a_\nu \Gamma_{\mu\lambda}^\nu E_b^\lambda = \frac{\kappa}{2} (\partial_b h_\mu^a - \partial^a h_{b\mu}) + O(\kappa^2) \quad (6.33)$$

and of the Ricci scalar Eq.(6.24), we see that our result (6.30) is indeed the linearization of the exact expression (6.32).

Adding last but not least the Bardeen-Zumino polynomial  $\mathcal{P}_{\mu\nu}$

$$\mathcal{P}_\mu^\mu = \pm \frac{1}{96\pi} \varepsilon^{ab} \nabla^\mu \omega_{ab\mu} \quad (6.34)$$

with its linearization

$$\mathcal{P}_\mu^\mu = \pm \frac{1}{96\pi} \varepsilon^{ab} \partial^\mu \partial_b h_{\mu a} \quad (6.35)$$

we find for the covariant trace anomaly

$$\langle T_\mu^\mu \rangle = \frac{1}{48\pi} \mathcal{R} \quad (6.36)$$

and

$$\langle T_\mu^\mu \rangle = \frac{1}{48\pi} (\partial_\mu \partial_\nu h^{\mu\nu} - \partial_\mu \partial^\mu h^\nu_\nu) \quad (6.37)$$

for its linearized version. Clearly these results are in agreement with Ref. [22].

## VII. SCHWINGER TERMS

In quantum field theory the energy-momentum tensors form an algebra which is generally not closed but has central extensions, so-called Schwinger terms, for example

$$[T_{00}(x), T_{00}(0)]_{ET} = i(T_{01}(x) + T_{01}(0)) \partial_1 \delta(x^1) + S_{0000} \quad (7.1)$$

$$[T_{01}(x), T_{01}(0)]_{ET} = i(T_{01}(x) + T_{01}(0)) \partial_1 \delta(x^1) + S_{0101} \quad (7.2)$$

$$[T_{00}(x), T_{01}(0)]_{ET} = i(T_{00}(x) + T_{00}(0)) \partial_1 \delta(x^1) + S_{0001} . \quad (7.3)$$

The Schwinger terms, the c-number terms  $S_{0000}, S_{0101}, S_{0001}$ , can be determined by considering the vacuum expectation value of the ETC.

To evaluate the ST we work with a technique that has been introduced by Källén [68] and is closely related to the dispersive approach used before. This technique has been applied already by Sýkora [40] to compute the ST for currents in Yang-Mills theories. Our aim is to generalize this procedure to the case of gravitation, where the current is replaced by the energy-momentum tensor.

From the Lagrangian (2.1) describing a Weyl fermion in a gravitational background field in two dimensions we get the following classical energy-momentum tensor

$$T_{\mu\nu} = : \frac{i}{4} \bar{\psi} \left( \gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu \right) P_\pm \psi : = \frac{1}{2} (T_{\mu\nu}^V \pm T_{\mu\nu}^A) , \quad (7.4)$$

where  $::$  means normal ordering. Using relation (2.29) and the equations of motions we can express the pseudo tensor part of the energy-momentum tensor by the pure tensor part (recall that the tensor is symmetric)

$$T_{\mu\nu}^A = -\varepsilon_\mu^\lambda T_{\lambda\nu}^V . \quad (7.5)$$

In two dimensions we have the identity

$$\varepsilon_\mu^\lambda \varepsilon_\rho^\tau = -g_{\mu\rho} g^{\lambda\tau} + g_\rho^\lambda g_\mu^\tau , \quad (7.6)$$

so that we find

$$\begin{aligned} \langle 0 | [T_{\mu\nu}(x), T_{\rho\sigma}(0)] | 0 \rangle &= \frac{1}{4} \left\{ \langle 0 | [T_{\mu\nu}^V(x), T_{\rho\sigma}^V(0)] | 0 \rangle + \langle 0 | [T_{\rho\nu}^V(x), T_{\mu\sigma}^V(0)] | 0 \rangle \right. \\ &\quad \left. - g_{\mu\rho} \langle 0 | [T_\nu^{V\lambda}(x), T_{\lambda\sigma}^V(0)] | 0 \rangle \mp \varepsilon_\mu^\lambda \langle 0 | [T_{\lambda\nu}^V(x), T_{\rho\sigma}^V(0)] | 0 \rangle \mp \varepsilon_\rho^\lambda \langle 0 | [T_{\mu\nu}^V(x), T_{\lambda\sigma}^V(0)] | 0 \rangle \right\} . \end{aligned} \quad (7.7)$$

Let us define

$$F_{\mu\nu\rho\sigma}(x) := \langle 0 | T_{\mu\nu}^V(x) T_{\rho\sigma}^V(0) | 0 \rangle . \quad (7.8)$$

By inserting the completeness relations  $\sum_n |n\rangle \langle n| = \mathbf{1}$  and using the translation invariance

$$T_{\mu\nu}^V(x) = e^{iPx} T_{\mu\nu}^V(0) e^{-iPx} , \quad (7.9)$$

we obtain

$$\begin{aligned}
F_{\mu\nu\rho\sigma}(x) &= \sum_n \langle 0|T_{\mu\nu}^V(x)|n\rangle \langle n|T_{\rho\sigma}^V(0)|0\rangle \\
&= \sum_n \langle 0|T_{\mu\nu}^V(0)|n\rangle \langle n|T_{\rho\sigma}^V(0)|0\rangle e^{-ip_n x},
\end{aligned} \tag{7.10}$$

where the sum runs over many-particle states  $|n\rangle$  with positive energy and momentum  $p_n$ . We may write

$$F_{\mu\nu\rho\sigma}(x) = \int d^2p \, e^{-ipx} G_{\mu\nu\rho\sigma}(p) \theta(p^0), \tag{7.11}$$

where

$$G_{\mu\nu\rho\sigma}(p) = \sum_n \delta(p_n - p) \langle 0|T_{\mu\nu}^V(0)|n\rangle \langle n|T_{\rho\sigma}^V(0)|0\rangle. \tag{7.12}$$

From Lorentz covariance and symmetry we get the same decomposition into formfactors as for  $T_{\mu\nu\rho\sigma}^V$  (see Eq. (2.15))

$$\begin{aligned}
G_{\mu\nu\rho\sigma}(p) &= p_\mu p_\nu p_\rho p_\sigma G_1(p^2) + (p_\mu p_\nu g_{\rho\sigma} + p_\rho p_\sigma g_{\mu\nu}) G_2(p^2) \\
&\quad + (p_\mu p_\rho g_{\nu\sigma} + p_\mu p_\sigma g_{\nu\rho} + p_\nu p_\rho g_{\mu\sigma} + p_\nu p_\sigma g_{\mu\rho}) G_3(p^2) \\
&\quad + g_{\mu\nu} g_{\rho\sigma} G_4(p^2) + (g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) G_5(p^2).
\end{aligned} \tag{7.13}$$

Making use of  $\partial^\mu T_{\mu\nu}^V(x) = 0$  provides the Ward identity  $p^\mu G_{\mu\nu\rho\sigma}(p) = 0$  that can be expressed by the formfactors

$$p^2 G_1 + G_2 + 2G_3 = 0 \tag{7.14}$$

$$p^2 G_2 + G_4 = 0 \tag{7.15}$$

$$p^2 G_3 + G_5 = 0. \tag{7.16}$$

Now let us explicitly evaluate  $G_{\mu\nu\rho\sigma}(p)$ . As we are considering the energy-momentum tensor as a free (non interacting) tensor – analogously to the case of free currents – we only need to sum up states that consist of one fermion-antifermion pair in Eq. (7.12). We get

$$G_{\mu\nu\rho\sigma}(p) = \int dp_1 \int dp_2 \sum_{s_1} \sum_{s_2} \delta(p - p_1 - p_2) \langle 0|T_{\mu\nu}^V(0)|n\rangle \langle n|T_{\rho\sigma}^V(0)|0\rangle, \tag{7.17}$$

where

$$|n\rangle = b^{\dagger(s_1)}(p_1) d^{\dagger(s_2)}(p_2) |0\rangle. \tag{7.18}$$

Let us assume that the fermion fields are canonically quantized

$$\psi(x) = \frac{1}{(2\pi)^{1/2}} \int dp \sum_{s=1}^2 \sqrt{\frac{m}{E_p}} [b^{(s)}(p) u^{(s)}(p) e^{-ipx} + d^{\dagger(s)}(p) v^{(s)}(p) e^{ipx}] \tag{7.19}$$

$$\bar{\psi}(x) = \frac{1}{(2\pi)^{1/2}} \int dp \sum_{s=1}^2 \sqrt{\frac{m}{E_p}} [b^{\dagger(s)}(p) \bar{u}^{(s)}(p) e^{ipx} + d^{(s)}(p) \bar{v}^{(s)}(p) e^{-ipx}], \tag{7.20}$$

then we find

$$\langle 0|T_{\mu\nu}^V(0)|n\rangle = \frac{1}{8\pi} \frac{m}{\sqrt{E_{p_1}E_{p_2}}} \bar{v}^{(s_2)}(p_2) (\gamma_\mu(p_1 - p_2)_\nu + \gamma_\nu(p_1 - p_2)_\mu) u^{(s_1)}(p_1) \quad (7.21)$$

$$\langle n|T_{\rho\sigma}^V(0)|0\rangle = -\frac{1}{8\pi} \frac{m}{\sqrt{E_{p_1}E_{p_2}}} \bar{u}^{(s_1)}(p_1) (\gamma_\rho(p_1 - p_2)_\sigma + \gamma_\sigma(p_1 - p_2)_\rho) v^{(s_2)}(p_2). \quad (7.22)$$

This provides

$$\begin{aligned} G_{\mu\nu\rho\sigma}(p) = & -\frac{1}{64\pi^2} \int dp_1 \int dp_2 \sum_{s_1} \sum_{s_2} \delta(p - p_1 - p_2) \frac{m^2}{E_{p_1}E_{p_2}} \\ & \times (p_1 - p_2)_\nu (p_1 - p_2)_\sigma \bar{v}^{(s_2)}(p_2) \gamma_\mu u^{(s_1)}(p_1) \bar{u}^{(s_1)}(p_1) \gamma_\rho v^{(s_2)}(p_2) \\ & + (\mu \leftrightarrow \nu) + (\rho \leftrightarrow \sigma) + \begin{pmatrix} \mu \leftrightarrow \nu \\ \rho \leftrightarrow \sigma \end{pmatrix}. \end{aligned} \quad (7.23)$$

Without the interchanges we call this  $G_{\mu\nu\rho\sigma}^{ni}(p)$ .

If we use the completeness relations

$$\sum_{s=1}^2 u^{(s)}(p) \bar{u}^{(s)}(p) = \frac{\not{p} + m}{2m} \quad (7.24)$$

$$\sum_{s=1}^2 v^{(s)}(p) \bar{v}^{(s)}(p) = \frac{\not{p} - m}{2m} \quad (7.25)$$

and the integral equation

$$\int \frac{dp}{2E_p} f(p) = \int d^2p \delta(p^2 - m^2) \theta(p^0) f(p), \quad (7.26)$$

we obtain

$$\begin{aligned} G_{\mu\nu\rho\sigma}^{ni} = & -\frac{1}{64\pi^2} \int d^2p_1 d^2p_2 \delta(p - p_1 - p_2) \delta(p_1^2 - m^2) \delta(p_2^2 - m^2) \theta(p_1^0) \theta(p_2^0) \\ & \times (p_1 - p_2)_\nu (p_1 - p_2)_\sigma \text{tr} \left[ \gamma_\mu (\not{p}_1 + m) \gamma_\rho (\not{p}_2 - m) \right]. \end{aligned} \quad (7.27)$$

Integrating next over the first  $\delta$ -function and evaluating the trace gives

$$\begin{aligned} G_{\mu\nu\rho\sigma}^{ni} = & -\frac{1}{32\pi^2} \int d^2p_1 \delta(p_1^2 - m^2) \delta((p - p_1)^2 - m^2) \theta(p_1^0) \theta(p^0 - p_1^0) \\ & \times (2p_1 - p)_\nu (2p_1 - p)_\sigma \left[ p_{1\mu} (p - p_1)_\rho + p_{1\rho} (p - p_1)_\mu \right. \\ & \left. - g_{\mu\rho} p_1^\lambda (p - p_1)_\lambda - g_{\mu\rho} m^2 \right]. \end{aligned} \quad (7.28)$$

If we compare this with (3.1) and (3.2) ( $p_1 = -k$ ), we see that  $G_{\mu\nu\rho\sigma}(p)$  is given by the imaginary part of the amplitude of the pure vector loop



$$G_{\mu\nu\rho\sigma}(p) = \frac{1}{2\pi^2} I m T_{\mu\nu\rho\sigma}^{pv}(p). \quad (7.29)$$

This is the important relation which links the Schwinger terms to the gravitational anomalies.

From Eq. (3.14) – (3.18) we read off the formfactors

$$G_1(p^2) = -\frac{1}{4\pi^2} J_0 \frac{m^2}{p^2} \left( 1 - 4 \frac{m^2}{p^2} \right) \quad (7.30)$$

$$G_2(p^2) = -\frac{1}{48\pi^2} J_0 p^2 \left( 1 - 8 \frac{m^2}{p^2} + 16 \frac{m^4}{p^4} \right) \quad (7.31)$$

$$G_3(p^2) = \frac{1}{96\pi^2} J_0 p^2 \left( 1 + 4 \frac{m^2}{p^2} - 32 \frac{m^4}{p^4} \right) \quad (7.32)$$

$$G_4(p^2) = \frac{1}{48\pi^2} J_0 p^4 \left( 1 - 8 \frac{m^2}{p^2} + 16 \frac{m^4}{p^4} \right) \quad (7.33)$$

$$G_5(p^2) = -\frac{1}{96\pi^2} J_0 p^4 \left( 1 + 4 \frac{m^2}{p^2} - 32 \frac{m^4}{p^4} \right). \quad (7.34)$$

Now we consider the commutator

$$\langle 0 | [T_{\mu\nu}^V(x), T_{\rho\sigma}^V(0)] | 0 \rangle = F_{\mu\nu\rho\sigma}(x) - F_{\rho\sigma\mu\nu}(-x) = \int d^2p \, e^{-ipx} \varepsilon(p^0) G_{\mu\nu\rho\sigma}(p). \quad (7.35)$$

If we remove the mass,  $m \rightarrow 0$ , that acted as an infrared cutoff, we get

$$G_1(p^2) = \lim_{m \rightarrow 0} -\frac{1}{4\pi^2} \frac{m^2}{p^4} \left( 1 - \frac{4m^2}{p^2} \right)^{\frac{1}{2}} \theta(p^2 - 4m^2) = -\frac{1}{24\pi^2} \delta(p^2). \quad (7.36)$$

From Eq. (7.35) we explicitly find

$$\langle 0 | [T_{00}^V(x), T_{00}^V(0)] | 0 \rangle_{ET} = \lim_{x_0 \rightarrow 0} -\frac{1}{24\pi^2} \int d^2p \, e^{-ipx} p_1^4 \varepsilon(p^0) \delta(p^2) = 0 \quad (7.37)$$

$$\langle 0 | [T_{11}^V(x), T_{11}^V(0)] | 0 \rangle_{ET} = \lim_{x_0 \rightarrow 0} -\frac{1}{24\pi^2} \int d^2p \, e^{-ipx} p_0^4 \varepsilon(p^0) \delta(p^2) = 0 \quad (7.38)$$

$$\begin{aligned} \langle 0 | [T_{00}^V(x), T_{11}^V(0)] | 0 \rangle_{ET} &= \langle 0 | [T_{01}^V(x), T_{01}^V(0)] | 0 \rangle_{ET} \\ &= \lim_{x_0 \rightarrow 0} -\frac{1}{24\pi^2} \int d^2p \, e^{-ipx} p_0^2 p_1^2 \varepsilon(p^0) \delta(p^2) = 0 \end{aligned} \quad (7.39)$$

$$\langle 0 | [T_{00}^V(x), T_{01}^V(0)] | 0 \rangle_{ET} = \lim_{x_0 \rightarrow 0} -\frac{1}{24\pi^2} \int d^2p \, e^{-ipx} p_0 p_1^3 \varepsilon(p^0) \delta(p^2) \quad (7.40)$$

$$\langle 0 | [T_{01}^V(x), T_{11}^V(0)] | 0 \rangle_{ET} = \lim_{x_0 \rightarrow 0} -\frac{1}{24\pi^2} \int d^2p \, e^{-ipx} p_0^3 p_1 \varepsilon(p^0) \delta(p^2). \quad (7.41)$$

The first three expressions vanish because  $\varepsilon(p^0)$  is antisymmetric. To evaluate the next two we use the Pauli-Jordan function

$$\Delta(x) = \frac{1}{2\pi} \int d^2p \, e^{-ipx} \varepsilon(p^0) \delta(p^2) \quad (7.42)$$

with the properties

$$\partial^\mu \partial_\mu \Delta(x) = 0 \quad (7.43)$$

$$\Delta(x)|_{x^0=0} = 0 \quad (7.44)$$

$$\partial_0 \Delta(x)|_{x^0=0} = -i\delta(x^1) \quad (7.45)$$

and we find

$$\lim_{x_0 \rightarrow 0} -\frac{1}{24\pi^2} \int d^2p \, e^{-ipx} \varepsilon(p^0) p_0 p_1 p^2 \delta(p^2) = \lim_{x_0 \rightarrow 0} -\frac{1}{12\pi} \partial_0 \partial_1 (\partial_0^2 - \partial_1^2) \Delta(x) = 0. \quad (7.46)$$

So we conclude

$$\begin{aligned} \langle 0|[T_{00}^V(x), T_{01}^V(0)]|0\rangle_{ET} &= \langle 0|[T_{01}^V(x), T_{11}^V(0)]|0\rangle_{ET} \\ &= \lim_{x_0 \rightarrow 0} -\frac{1}{12\pi} \partial_1^3 \partial_0 \Delta(x) = \frac{i}{12\pi} \partial_1^3 \delta(x^1). \end{aligned} \quad (7.47)$$

With relation (7.7) we finally obtain the Schwinger terms in the ETC of the energy-momentum tensors

$$\langle 0|[T_{00}(x), T_{00}(0)]|0\rangle_{ET} = \langle 0|[T_{11}(x), T_{11}(0)]|0\rangle_{ET} = \mp \frac{i}{24\pi} (\partial_1)^3 \delta(x^1) \quad (7.48)$$

$$\langle 0|[T_{00}(x), T_{11}(0)]|0\rangle_{ET} = \langle 0|[T_{01}(x), T_{01}(0)]|0\rangle_{ET} = \mp \frac{i}{24\pi} (\partial_1)^3 \delta(x^1) \quad (7.49)$$

$$\langle 0|[T_{00}(x), T_{01}(0)]|0\rangle_{ET} = \langle 0|[T_{01}(x), T_{11}(0)]|0\rangle_{ET} = \frac{i}{24\pi} (\partial_1)^3 \delta(x^1). \quad (7.50)$$

Our result agrees with the one of Tomiya [52] who works with an invariant spectral function method and in addition uses a cohomological approach. It also coincides with the result of Ebner, Heid and Lopes-Cardoso [51] who derive the Schwinger terms directly from the gravitational anomaly. In our approach Eq. (7.29) is the basic relation. It connects the Schwinger terms, determined by  $G_{\mu\nu\rho\sigma}$ , with the (linearized) gravitational anomalies given by  $Im T_{\mu\nu\rho\sigma}^{pv}$  in our dispersion relations procedure.

## VIII. CONCLUSIONS

We have investigated the gravitational anomalies, specifically the pure Einstein anomaly and the Weyl anomaly. So we demanded the quantized energy-momentum tensor to be symmetric – no Lorentz anomaly occurs – which is a possible choice. The relevant amplitude, the two-point function of the energy-momentum tensors  $T_{\mu\nu\rho\sigma}$ , we have separated into its pure tensor part  $T_{\mu\nu\rho\sigma}^V$  and into its pseudo-tensor part  $T_{\mu\nu\rho\sigma}^A$ , Eqs.(2.13) – (2.16), and we have decomposed the amplitudes into a general structure of tensors containing 8 formfactors  $T_1(p^2), \dots, T_8(p^2)$ .

These formfactors we have expressed by dispersion relations where we had to calculate only the imaginary parts via the Cutkosky rules. Our subtraction procedure for the formfactors – no subtraction for  $T_1, T_6$ , one subtraction for  $T_2, T_3, T_7, T_8$  and two subtractions for  $T_4, T_5$  – is *the* 'natural' choice dictated by the  $t$ -behaviour of the imaginary parts  $Im T_i(t)$ .

It implies that the pure tensor WI (2.21) – (2.23) for the renormalized formfactors is satisfied (in the limit  $m \rightarrow 0$ ), so that the total anomaly is automatically shifted into the pseudotensor part of the WI (2.18). It turns out that the anomalous Ward identity and the anomalous trace identity depend only on the finite formfactor  $T_1 = \mp 4T_6$ , with its explicit result (4.31), demonstrating such the independence of a special renormalization procedure. From the anomalous Ward identity and the anomalous trace identity we could deduce the linearized gravitational anomalies – the linearized Einstein- and Weyl anomaly – and determine their covariant versions.

The origin of the anomaly is the existence of a superconvergence sum rule for the imaginary part of the formfactor  $T_1(p^2)$ . In the zero mass limit the imaginary part of the formfactor approaches a  $\delta$ -function singularity at zero momentum squared, exhibiting in this way the infrared feature of the gravitational anomalies. This is an independent and complementary view of the anomalies as compared to the ultraviolet regularization procedures. If we compare, however, the DR approach with the  $n$ -dimensional regularization procedure of 't Hooft–Veltman we find an equivalence. The two approaches are linked by the substitutions (5.1).

We have also calculated the gravitational Schwinger terms which occur in the ETC of the energy-momentum tensors. We have adapted our dispersive approach to the method of Källén. As a result all gravitational Schwinger terms are determined by the formfactor  $G_1(p^2)$  which in the zero mass limit approaches a  $\delta$ -function singularity at zero momentum squared – as in the case of anomalies. So also the Schwinger terms show this peculiar infrared feature of the anomalies.

We have performed all calculations in two dimensions, where already all essential features of the DR approach show up, analogously to the chiral current case. From a practical point of view the method appears quite appealing. All one has to calculate is the imaginary part of an amplitude (formfactor), which is an easy task. However, this computational simplicity is a special (and convenient) feature of the two space-time dimensions. In higher dimensions the amplitude will contain more formfactors and we have to calculate dispersion relations for higher loop diagrams, which is a much more delicate task, but nevertheless we expect the method to work here too.

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